Existence of weak solutions for mean curvature flow with a non-local term (Analysis on Shapes of Solutions to Partial Differential Equations)

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1 Introduction

Let $U_t \subset \mathbb{R}^d$ be a bounded open set and have a smooth boundary M_t for $t \in [0,T)$. The family of hypersurfaces $\{M_t\}_{t\in[0,T)}$ is called the volume preserving mean curvature flow if the velocity vector v of M_t is given by

$$v = h - \langle h \cdot n\rangle n \quad \text{on } M_t,\ t \in (0,T), \tag{1.1}$$

where h and n are the mean curvature vector and the inner unit normal vector of M_t respectively, and

$$\langle h \cdot n\rangle := \frac{1}{\mathcal{H}^{d-1}(M_t)} \int_{M_t} h \cdot n \, d\mathcal{H}^{d-1}.$$

Here $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. By (1.1), for the volume preserving mean curvature flow $\{M_t\}_{t\in[0,T)}$ we have

$$\frac{d}{dt}\mathcal{L}^d(U_t) = -\int_{M_t} v \cdot n \, d\mathcal{H}^{d-1} = 0 \quad \text{(volume preserving property)}, \tag{1.2}$$

where $\mathcal{L}^d$ is the d-dimensional Lebesgue measure. By (1.2) we have

$$\begin{aligned} \frac{d}{dt}\mathcal{H}^{d-1}(M_t) &= -\int_{M_t} h \cdot v \, d\mathcal{H}^{d-1} = -\int_{M_t} (v + \langle h \cdot n\rangle n) \cdot v \, d\mathcal{H}^{d-1} \\ &= -\int_{M_t} |v|^2 \, d\mathcal{H}^{d-1} \le 0 \end{aligned} \tag{1.3}$$

for the solution for (1.1).

The time global existence of the classical solution to (1.1) for convex initial data was proved by Gage [8] ($d = 2$) and Huisken [10] ($d \ge 2$). Escher and Simonett [7] proved the short time existence of the solution to (1.1) for smooth initial data, and they showed that if M_0 is sufficiently close to the Euclidean sphere, then there exists

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the time global solution. Mugnai, Seis and Spadaro [15] proved the existence of the global distributional solution to (1.1) by using a variational approach. Takasao [19] showed the weak solution to (1.1) via the phase field method for $d = 2, 3$.

Let $g = g(x,t)$ be a smooth function with $g(x,t) > 0$ for any $(x,t) \in \mathbb{R}^d \times [0,\infty)$. In this article, we consider the following weighted volume preserving mean curvature flow equation:

$$v = h - \Big(\frac{\int_{M_t}(h\cdot n)g\,d\mathcal{H}^{d-1} - \int_{U_t}\partial_t g\,dx}{\int_{M_t} g^2\,d\mathcal{H}^{d-1}}\Big)gn \quad \text{on } M_t,\ t\in(0,T). \tag{1.4}$$

Note that if g is constant then (1.4) is the volume preserving mean curvature flow equation (1.1). For the solution $\{M_t\}_{t\in[0,T)}$ of (1.4), we have the weighted volume preserving property:

$$\frac{d}{dt}\int_{U_t} g\,dx = -\int_{M_t}(v\cdot n)g\,d\mathcal{H}^{d-1} + \int_{U_t}\partial_t g\,dx = 0. \tag{1.5}$$

Set $\Lambda = \Lambda(t) := \frac{\int_{M_t}(h\cdot n)g\,d\mathcal{H}^{d-1} - \int_{U_t}\partial_t g\,dx}{\int_{M_t} g^2\,d\mathcal{H}^{d-1}}$. By (1.4) and (1.5) we have

$$\begin{aligned}\frac{d}{dt}\mathcal{H}^{d-1}(M_t) &= -\int_{M_t} h\cdot v\,d\mathcal{H}^{d-1} = -\int_{M_t}(v+\Lambda gn)\cdot v\,d\mathcal{H}^{d-1}\\ &= -\int_{M_t}|v|^2\,d\mathcal{H}^{d-1} - \Lambda\int_{M_t}(v\cdot n)g\,d\mathcal{H}^{d-1}\\ &= -\int_{M_t}|v|^2\,d\mathcal{H}^{d-1} - \Lambda\int_{U_t}\partial_t g\,dx.\end{aligned} \tag{1.6}$$

Hence, if g depends only on x, then we obtain

$$\frac{d}{dt}\mathcal{H}^{d-1}(M_t) = -\int_{M_t}|v|^2\,d\mathcal{H}^{d-1} \le 0. \tag{1.7}$$

Harthley also studied the following another weighted volume preserving mean curvature flow equation:

$$v = h - \Big(\frac{\int_{M_t}(h\cdot n)\tilde{g}\,d\mathcal{H}^{d-1}}{\int_{M_t}\tilde{g}\,d\mathcal{H}^{d-1}}\Big)n \quad \text{on } M_t,\ t\in(0,T), \tag{1.8}$$

where $\tilde{g} = \tilde{g}(x)$ is a given function with $\tilde{g}(x) > 0$ for any $x \in \mathbb{R}^d$. We remark that for the solution $\{M_t\}_{t\in[0,T)}$ of (1.8) we also have the weighted volume preserving property:

$$\frac{d}{dt}\int_{U_t}\tilde{g}\,dx = -\int_{M_t}(v\cdot n)\tilde{g}\,d\mathcal{H}^{d-1} = 0. \tag{1.9}$$

Remark 1.1. The solution for (1.8) does not satisfy (1.7) in general.

Remark 1.2. Set $M_t^r := \{x \in \mathbb{R}^d \mid |x| = r\}$ for $r > 0$. Then $\{M_t^r\}_{t\ge0}$ is a stationary solution for (1.8), even if $\tilde{g}$ is not a constant function. However, whether $\{M_t^r\}_{t\ge0}$ is a solution for (1.4) or not depends on g.

Let $U \subset \mathbb{R}^d$ be a bounded open set with smooth boundary $M = \partial U$. For any $f \in C_c^1(\mathbb{R}^d;\mathbb{R}^d)$, we define $U_\delta := \{y \in \mathbb{R}^d \,|\, y = x + \delta f(x),\ x \in U\}$ and $M_\delta := \{y \in \mathbb{R}^d \,|\, y = x + \delta f(x),\ x \in M\}$ for $\delta \in (-1,1)$. Then we have

$$\frac{d}{d\delta}\mathcal{H}^{d-1}(M_\delta)\Big|_{\delta=0} = -\int_M h\cdot f\, d\mathcal{H}^{d-1} \quad \text{and} \quad \frac{d}{d\delta}\mathcal{L}^d(U_\delta)\Big|_{\delta=0} = -\int_M n\cdot f\, d\mathcal{H}^{d-1}, \tag{1.10}$$

where h and n are the mean curvature vector and the inner unit normal vector of M respectively. Thus

$$\frac{d}{d\delta}\Big(\mathcal{H}^{d-1}(M_\delta) - \lambda\mathcal{L}^d(U_\delta)\Big)\Big|_{\delta=0} = -\int_M (h - \lambda n)\cdot f\, d\mathcal{H}^{d-1} \quad \text{for } \lambda \in \mathbb{R}. \tag{1.11}$$

Therefore (1.1) is a gradient flow for the area $\mathcal{H}^{d-1}(M_t)$ subject to $\mathcal{L}^d(U_t) = \mathcal{L}^d(U_0)$.

Let $g \in C(\mathbb{R}^d)$ be a positive function. By an argument similar to (1.10) and (1.11), we have

$$\frac{d}{d\delta}\int_{U_\delta} g(x)\, dx\Big|_{\delta=0} = -\int_M g(n\cdot f)\, d\mathcal{H}^{d-1} \tag{1.12}$$

and

$$\frac{d}{d\delta}\Big(\mathcal{H}^{d-1}(M_\delta) - \lambda\int_{U_\delta} g(x)\, dx\Big)\Big|_{\delta=0} = -\int_M (h - \lambda g n)\cdot f\, d\mathcal{H}^{d-1}, \quad \text{for } \lambda \in \mathbb{R}. \tag{1.13}$$

Hence (1.4) is a gradient flow for the area $\mathcal{H}^{d-1}(M_t)$ subject to $\int_{U_t} g\, dx = \int_{U_t} g\, dx\Big|_{t=0}$.

2 Phase field methods for (1.1) and (1.4)

In this section, we first compare the two phase field methods for (1.1), and then we introduce the phase field method used for the proof of the existence of the weak solution for (1.4). For simplicity, we consider the periodic boundary condition from this section.

Let $\varepsilon \in (0,1)$ and $\Omega := \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. We also use Ω to a set $[0,1)^d$. To study (1.1), Rubinstein and Sternberg [17] considered the following non-local reaction diffusion equation:

$$\begin{cases} \varepsilon\partial_t\varphi^\varepsilon = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + \lambda_{RS}^\varepsilon, & (x,t) \in \Omega\times(0,\infty), \\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where $W(s) := (1-s^2)^2/2$ and $\lambda_{RS}^\varepsilon(t) := \frac{1}{|\Omega|}\int_\Omega \frac{W'(\varphi^\varepsilon)}{\varepsilon}\, dx$. Note that the solution φ^ε for (2.1) satisfies the following volume preserving property:

$$\frac{d}{dt}\int_\Omega \varphi^\varepsilon\, dx = 0. \tag{2.2}$$

Chen, Hilhorst and Logak [4] proved that the zero level set of the solution for (2.1) converges to the classical solution of (1.1) under several suitable conditions.

Remark 2.1. To obtain the existence of the weak solution for (1.1) via (2.1), we need the L^2-estimates of the mean curvature, that is,

$$\sup_{\varepsilon\in(0,1)}\int_0^T\int_\Omega \varepsilon\Big(-\Delta\varphi^\varepsilon+\frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)^2\,dxdt<\infty$$

(see Remark 3.8 and Theorem 4.3). However, whether the solutions for (2.1) have the estimates or not is an open problem, due to the difficulty of the estimates of λ^ε_{RS} (see Remark 2.6).

In 1997, Golovaty [9] studied the singular limit of the radially symmetric solutions for the following non-local reaction diffusion equation:

$$\begin{cases} \varepsilon\partial_t\varphi^\varepsilon=\varepsilon\Delta\varphi^\varepsilon-\dfrac{W'(\varphi^\varepsilon)}{\varepsilon}+\lambda^\varepsilon_G\sqrt{2W(\varphi^\varepsilon)}, & (x,t)\in\Omega\times(0,\infty),\\ \varphi^\varepsilon(x,0)=\varphi^\varepsilon_0(x), & x\in\Omega, \end{cases} \tag{2.3}$$

where

$$\lambda^\varepsilon_G=\lambda^\varepsilon_G(t):=\frac{\int_\Omega\sqrt{2W(\varphi^\varepsilon)}\Big(-\Delta\varphi^\varepsilon+\frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)\,dx}{2\int_\Omega\frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx}.$$

Takasao [19] proved the global existence of the weak solution for (1.1) via the singular limit of the solutions for (2.3).

Note that the solution φ^ε for (2.3) also satisfies the following volume preserving property:

$$\frac{d}{dt}\int_\Omega k(\varphi^\varepsilon)\,dx=\int_\Omega\sqrt{2W(\varphi^\varepsilon)}\partial_t\varphi^\varepsilon\,dx=0, \tag{2.4}$$

where $k(s)=\int_0^s\sqrt{2W(\tau)}\,d\tau=-\frac13 s^3+s$. By the integration by parts, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2}+\frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)\,dx=\int_\Omega\Big(\varepsilon\nabla\varphi^\varepsilon\cdot\nabla\partial_t\varphi^\varepsilon+\frac{W'(\varphi^\varepsilon)}{\varepsilon}\partial_t\varphi^\varepsilon\Big)\,dx\\ &=\int_\Omega\Big(-\varepsilon\Delta\varphi^\varepsilon+\frac{W'(\varphi^\varepsilon)}{\varepsilon}\Big)\partial_t\varphi^\varepsilon\,dx=\int_\Omega(-\varepsilon\partial_t\varphi^\varepsilon+\lambda^\varepsilon\sqrt{2W(\varphi^\varepsilon)})\partial_t\varphi^\varepsilon\,dx\\ &=-\int_\Omega\varepsilon(\partial_t\varphi^\varepsilon)^2\,dx+\lambda^\varepsilon\int_\Omega\partial_t\varphi^\varepsilon\sqrt{2W(\varphi^\varepsilon)}\,dx=-\int_\Omega\varepsilon(\partial_t\varphi^\varepsilon)^2\,dx, \end{aligned} \tag{2.5}$$

where (2.4) is used. Note that (2.5) corresponds to (1.3).

Remark 2.2. Assume that $\varphi^\varepsilon\to\varphi=\pm1$ as $\varepsilon\to0$ for a.e. $(x,t)\in\Omega\times(0,\infty)$. Then by $k(\pm1)=\pm\frac23$ we have

$$\lim_{\varepsilon\to0}\int_\Omega k(\varphi^\varepsilon)\,dx=\frac23\int_\Omega\varphi\,dx.$$

Thus by (2.4) we obtain the volume preserving property:

$$\int_\Omega\varphi(x,t)\,dx=\int_\Omega\varphi_0\,dx\quad\text{for } t\geq0.$$

Remark 2.3. For $\varphi \in C^2(\Omega)$, we define

$$E(\varphi) := \int_\Omega \Big(\frac{\varepsilon |\nabla \varphi|^2}{2} + \frac{W(\varphi)}{\varepsilon} \Big)\, dx \quad \text{and} \quad F(\varphi) := \int_\Omega k(\varphi)\, dx.$$

Then, for $\psi \in C^1(\Omega)$ we have

$$\frac{d}{d\delta} E(\varphi + \delta\psi)\Big|_{\delta=0} = \int_\Omega \Big(-\varepsilon \Delta \varphi + \frac{W'(\varphi)}{\varepsilon} \Big) \psi\, dx$$

and

$$\frac{d}{d\delta} F(\varphi + \delta\psi)\Big|_{\delta=0} = \int_\Omega \sqrt{2W(\varphi)} \psi\, dx.$$

Therefore (2.3) is the gradient flow for $E(\varphi^\varepsilon)$ subject to $\int_\Omega k(\varphi^\varepsilon(x,t))\, dx = \int_\Omega k(\varphi_0^\varepsilon)\, dx$.

Remark 2.4. By $\sqrt{2W(0)} = 1$ and $\sqrt{2W(\pm 1)} = 0$, it can be interpreted that the term $\lambda_G^\varepsilon \sqrt{2W(\varphi^\varepsilon)}$ of (2.3) affects φ^ε only on the neighborhood of the zero level set of φ^ε.

Remark 2.5. We denote $\sigma := \int_{-1}^1 \sqrt{2W(s)}\, ds$. The solution for the Allen-Cahn equation such as (2.1) and (2.3) has the following approximate expressions (see [11]):

$$\mathcal{H}^{d-1}(M_t^\varepsilon) \approx \frac{1}{\sigma} \int_\Omega \Big(\frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon} \Big)\, dx \tag{2.6}$$

and

$$\int_{M_t^\varepsilon} h^\varepsilon \cdot f\, d\mathcal{H}^{d-1} \approx \frac{1}{\sigma} \int_\Omega \varepsilon \Big(-\Delta \varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} \Big) \nabla \varphi^\varepsilon \cdot f\, dx, \tag{2.7}$$

where $M_t^\varepsilon := \{x \mid \varphi^\varepsilon(x,t) = 0\}$ and h^ε is the mean curvature vector for M_t^ε.

Assume that $\{M_t\}_{t\in[0,\infty)}$ is the solution for (1.1), $M_t^\varepsilon \approx M_t$ for sufficiently small $\varepsilon > 0$, and the following equilibrium of energy:

$$\frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2} \approx \frac{W(\varphi^\varepsilon)}{\varepsilon} \qquad \text{in } \Omega \times (0,\infty) \tag{2.8}$$

for the solution of (2.3) (see Theorem 4.3). Then we have

$$\frac{2}{\sigma} \int_\Omega \frac{W(\varphi^\varepsilon)}{\varepsilon}\, dx \approx \frac{1}{\sigma} \int_\Omega \Big(\frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon} \Big)\, dx \approx \mathcal{H}^{d-1}(M_t) \tag{2.9}$$

and

$$\begin{aligned} &\frac{1}{\sigma} \int_\Omega \sqrt{2W(\varphi^\varepsilon)} \Big(-\Delta \varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} \Big)\, dx \\ \approx &\frac{1}{\sigma} \int_\Omega \varepsilon \Big(-\Delta \varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} \Big) \nabla \varphi^\varepsilon \cdot n^\varepsilon\, dx \approx \int_{M_t} h \cdot n\, d\mathcal{H}^{d-1}. \end{aligned} \tag{2.10}$$

Here $n^\varepsilon := \frac{\nabla\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}$ is the inner unit normal vector of $\partial\{x \,|\, \varphi^\varepsilon(x,t) > 0\}$. by (2.9) and (2.10) we have

$$\lambda_G^\varepsilon = \frac{\int_\Omega \sqrt{2W(\varphi^\varepsilon)}\Big(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)\,dx}{2\int_\Omega \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx} \approx \frac{1}{\mathcal{H}^{d-1}(M_t)}\int_{M_t} h\cdot n\,d\mathcal{H}^{d-1}. \tag{2.11}$$

Hence, λ_G^ε seems complicated, however it is an approximation of the non-local term of (1.1).

Remark 2.6. Assume that there exist $D_0 > 0$ and $\omega_0 > 0$ such that

$$E(\varphi_0^\varepsilon) \le D_0$$

and

$$\Big|\int_\Omega k(\varphi_0^\varepsilon)\,dx\Big| \le \frac{2}{3} - \omega \tag{2.12}$$

for any $\varepsilon \in (0,1)$. Takasao [19] proved that there exist $\epsilon \in (0,1)$ and $C_0 > 0$ such that

$$\sup_{\varepsilon\in(0,\epsilon_0)} \int_0^T (\lambda_G^\varepsilon)^2\,dt \le C_0(1+T) \tag{2.13}$$

by using an argument similar to that in [3]. By (2.5) we have

$$E(\varphi^\varepsilon(\cdot,T)) + \int_0^T\int_\Omega \varepsilon(\partial_t\varphi^\varepsilon)^2\,dxdt = E(\varphi_0^\varepsilon) \le D_0 \tag{2.14}$$

for any $T > 0$. By (2.13) and (2.14) we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \varepsilon\Big(\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)^2\,dxdt\\
\le&\int_0^T\int_\Omega \varepsilon(\partial_t\varphi^\varepsilon)^2\,dxdt + \int_0^T\int_\Omega \varepsilon\Big(\lambda_G^\varepsilon\frac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon}\Big)^2\,dxdt\\
\le&D_0 + \int_0^T(\lambda_G^\varepsilon)^2\int_\Omega \frac{2W(\varphi^\varepsilon)}{\varepsilon}\,dxdt \le D_0 + 2D_0\int_0^T(\lambda_G^\varepsilon)^2dt\\
\le&D_0(1+2C_0(1+T)),
\end{aligned} \tag{2.15}$$

where $\int_\Omega \frac{2W(\varphi^\varepsilon)}{\varepsilon}\,dx \le 2E(\varphi^\varepsilon(\cdot,t)) \le 2E(\varphi_0^\varepsilon) \le 2D_0$ is used. Hence we obtain the L^2 estimate of the mean curvature(see Remark 2.1). For (2.1), Bronsard and Stoth [3] proved the boundedness of $\sup_\varepsilon \int_0^T (\lambda_{RS}^\varepsilon)^2\,dt$. However we need the boundedness of $\sup_\varepsilon \varepsilon^{-1}\int_0^T (\lambda_{RS}^\varepsilon)^2\,dt$ to obtain a estimate similar to (2.15).

For (1.4), we consider the following reaction diffusion equation:

$$\begin{cases} \varepsilon\partial_t\varphi^\varepsilon = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + \lambda^\varepsilon g\sqrt{2W(\varphi^\varepsilon)}, & (x,t)\in\Omega\times(0,\infty),\\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x\in\Omega, \end{cases} \tag{2.16}$$

where

$$\lambda^\varepsilon = \lambda^\varepsilon(t) := \frac{\int_\Omega \Big\{\sqrt{2W(\varphi^\varepsilon)}\Big(- \Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big) g - \tilde{k}(\varphi^\varepsilon)\partial_t g\Big\}\, dx}{2\int_\Omega g^2 \frac{W(\varphi^\varepsilon)}{\varepsilon}\, dx},$$

where $\tilde{k}(s) := k(s) + \int_0^1 \sqrt{2W(\tau)}\, d\tau = -\frac{1}{3}s^3 + s + \frac{2}{3}$. (2.16) has the following property:

$$\frac{d}{dt}\int_\Omega \tilde{k}(\varphi^\varepsilon) g\, dx = \int_\Omega \sqrt{2W(\varphi^\varepsilon)}\partial_t\varphi^\varepsilon g\, dx + \int_\Omega \tilde{k}(\varphi^\varepsilon)\partial_t g\, dx = 0. \tag{2.17}$$

By an argument similar to that in Remark 2.3, (2.16) is the gradient flow for $E(\varphi^\varepsilon)$ subject to $\int_\Omega k(\varphi^\varepsilon(x,t))g(x,t)\, dx = \int_\Omega k(\varphi_0^\varepsilon)g(x,0)\, dx$.

Remark 2.7. Assume that $\varphi^\varepsilon \to \varphi = \pm 1$ as $\varepsilon \to 0$ for a.e. $(x,t) \in \Omega \times (0,\infty)$. Then by $\tilde{k}(+1) = \frac{4}{3} = \sigma$ and $\tilde{k}(-1) = 0$, we have

$$\int_\Omega \tilde{k}(\varphi^\varepsilon) g\, dx \approx \int_\Omega \sigma\chi_{\{\varphi^\varepsilon \approx +1\}} g\, dx \approx \sigma\int_{U_t} g\, dx.$$

Therefore (2.17) corresponds to (1.5).

Remark 2.8. By an argument similar to (2.11), we have

$$\begin{aligned} \lambda^\varepsilon &= \frac{\sigma^{-1}\int_\Omega \Big\{\sqrt{2W(\varphi^\varepsilon)}\Big(- \Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big) g - \tilde{k}(\varphi^\varepsilon)\partial_t g\Big\}\, dx}{2\sigma^{-1}\int_\Omega g^2\frac{W(\varphi^\varepsilon)}{\varepsilon}\, dx} \\ &\approx \frac{\int_{M_t} (h\cdot n) g\, d\mathcal{H}^{d-1} - \int_{U_t}\partial_t g\, dx}{\int_{M_t} g^2\, d\mathcal{H}^{d-1}}. \end{aligned} \tag{2.18}$$

By (2.17) and the integration by parts, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)\, dx = \int_\Omega (-\varepsilon\partial_t\varphi^\varepsilon + \lambda^\varepsilon g\sqrt{2W(\varphi^\varepsilon)})\varphi_t^\varepsilon\, dx \\ &= -\int_\Omega \varepsilon(\partial_t\varphi^\varepsilon)^2\, dx + \lambda^\varepsilon\int_\Omega \partial_t\varphi^\varepsilon g\sqrt{2W(\varphi^\varepsilon)}\, dx \\ &= -\int_\Omega \varepsilon(\partial_t\varphi^\varepsilon)^2\, dx - \lambda^\varepsilon\int_\Omega \tilde{k}(\varphi^\varepsilon)\partial_t g\, dx. \end{aligned} \tag{2.19}$$

Note that (2.19) corresponds to (1.6), and if $\partial_t g \equiv 0$ then $\frac{d}{dt}E(\varphi^\varepsilon(\cdot,t)) \leq 0$.

3 Preliminaries and main results

In this section we define the weak solution (L^2-flow) and show the time global existence of the weak solution for (1.4). We recall some notations and definitions from geometric measure theory and refer to [1, 2, 5, 6, 18] for more details.

Let $d \geq k+1$ and $G_k(\mathbb{R}^d)$ be a Grassmann manifold of unoriented k-dimensional subspaces in $\mathbb{R}^d$.

Definition 3.1. A set $M \subset \mathbb{R}^d$ is called a countably k-rectifiable set if M is $\mathcal{H}^k$-measurable and there exists a family of C^1 k-dimensional embedded submanifolds $\{M_i\}_{i=1}^{\infty}$ such that $\mathcal{H}^k(M \setminus \cup_{i=1}^{\infty} M_i) = 0$.

Definition 3.2. Let M be an $\mathcal{H}^k$-measurable subset of $\mathbb{R}^d$ and $\theta \in L^1_{loc}(\mathcal{H}^k(M))$ is a positive function. We say M has an approximate tangent plane $P \in G_k(\mathbb{R}^d)$ at $x_0 \in M$ with respect to θ if

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x_0,\lambda}(M)} f(y)\theta(x_0+\lambda y)\, d\mathcal{H}^k(y) = \theta(x_0) \int_P f(y)\, d\mathcal{H}^k(y)$$

holds for any $f \in C_c(\mathbb{R}^d)$. Here $\eta_{x_0,\lambda}(x) := \frac{1}{\lambda}(x - x_0)$.

Remark 3.3. If $M \subset \mathbb{R}^d$ is $\mathcal{H}^k$-measurable and k-rectifiable, then there exists an approximate tangent plane with respect to θ $\mathcal{H}^k$-a.e. on M for any positive function $\theta \in L^1_{loc}(\mathcal{H}^k(M))$.

Definition 3.4. A Radon measure μ is called k-rectifiable if there exists a countable k-rectifiable set M and a function $\theta : M \to (0,\infty)$ such that $\theta \in L^1_{loc}(\mathcal{H}^k\lfloor_M)$ and $\mu = \theta\mathcal{H}^k\lfloor_M$, that is, $\mu(A) = \int_{A\cap M} \theta\, d\mathcal{H}^k$ for any measurable set $A \subset \mathbb{R}^d$. Moreover if $\theta \in \mathbb{N}$ $\mathcal{H}^k$-a.e. on M, μ is called k-integral.

Definition 3.5. Let M be an $\mathcal{H}^k$-measurable subset of $\mathbb{R}^d$ and $\theta \in L^1_{loc}(\mathcal{H}^k(M))$ is a positive function. For a $(d-1)$-rectifiable Radon measure $\mu = \theta\mathcal{H}^k\lfloor_M$, h is called a generalized mean curvature vector if

$$\int_{\mathbb{R}^d} \operatorname{div}_M g\, d\mu = -\int_{\mathbb{R}^d} h \cdot g\, d\mu \tag{3.1}$$

holds for any $g \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$. Here, $\operatorname{div}_M g = \sum_{k,l=1}^d \partial_{x_k} g_l(\delta_{kl} - \nu_k\nu_l)$, $\nu = (\nu_1,\dots,\nu_d)$ is the unit normal vector of the approximate tangent plane of M and $g = (g_1,\dots,g_d)$.

Remark 3.6. If $M \subset \mathbb{R}^d$ is an oriented smooth hypersurface, then by the divergence theorem for manifolds, we have

$$\int_M \operatorname{div}_M g\, d\mathcal{H}^{d-1} = -\int_M h\cdot g\, d\mathcal{H}^{d-1} + \int_{\partial M} \gamma \cdot g\, d\mathcal{H}^{d-2}$$

for any $g \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$, where h and γ are the mean curvature vector of M and the outer unit normal vector of M on ∂M, respectively. If $\partial M = \emptyset$, then h is also the generalized mean curvature vector with $\mu = \mathcal{H}^d\lfloor_M$ in (3.1).

The following definition is similar to Brakke's weak solution for the mean curvature flow:

Definition 3.7 (L^2-flow [13]). Let $U \subset \mathbb{R}^d$ be an open set and $\{\mu_t\}_{t\in(0,T)}$ be a family of Radon measures on U. We call $\{\mu_t\}_{t\in(0,T)}$ L^2-flow if the following hold:

1. μ_t is $(d-1)$-rectifiable and integral, and has a generalized mean curvature vector $h \in L^2(\mu_t)$ a.e. $t \in (0,T)$,

2. and there exists $C > 0$ and a vector field $v \in L^2(0,T;(L^2(\mu_t))^d)$ such that

$$\begin{cases} v(x,t) \perp T_x\mu_t \quad \text{for } \mu_t \otimes \mathcal{L}^1\text{-a.e. } (x,t) \in U \times (0,T), \\ \left| \int_0^T \int_U (\eta_t + \nabla\eta \cdot v)\, d\mu_t dt \right| \le C\|\eta\|_{C^0(U\times(0,T))} \quad \text{for any } \eta \in C_c^1(U \times (0,T)). \end{cases}$$

Here $T_x\mu_t$ is the approximate tangent plane of μ_t at x.

Moreover the vector valued function v is called a generalized velocity vector.

Remark 3.8. Let $M_t \subset U$ be a closed, bounded and smooth hypersurface for $t \in [0,T)$. Assume that $\{M_t\}_{t\in[0,T)}$ is a classical solution for the mean curvature flow equation with force term:

$$v = h + f \quad \text{on } M_t,\ t \in (0,T), \tag{3.2}$$

where f is a given smooth vector valued function with $\int_0^T \int_{M_t} |f|^2\, d\mathcal{H}^{d-1}dt \le C$ for some $C > 0$. Then we have

$$\begin{aligned} \mathcal{H}^{d-1}(M_T) - \mathcal{H}^{d-1}(M_0) &= \int_0^T \frac{d}{dt}\mathcal{H}^{d-1}(M_t)\, dt = -\int_0^T \int_{M_t} h \cdot v\, d\mathcal{H}^{d-1}dt \\ = -\int_0^T \int_{M_t} h \cdot (h+f)\, d\mathcal{H}^{d-1}dt &\le -\frac{1}{2}\int_0^T \int_{M_t} |h|^2\, d\mathcal{H}^{d-1}dt + \frac{1}{2}C. \end{aligned}$$

Hence we have the boundedness of $C_T := \left(\int_0^T \int_{M_t} |h|^2\, d\mathcal{H}^{d-1}dt\right)^{\frac{1}{2}} \left(\int_0^T \int_{M_t} |v|^2\, d\mathcal{H}^{d-1}dt\right)^{\frac{1}{2}}$. Set $\mu_t := \mathcal{H}^{d-1}\lfloor_{M_t}$. For any $\eta \in C_c^1(U \times (0,T))$ we compute that

$$\begin{aligned} &\left|\int_0^T \int_U (\partial_t\eta + \nabla\eta \cdot v)\, d\mu_t dt\right| = \left|\int_0^T \int_{M_t} (\partial_t\eta + \nabla\eta \cdot v)\, d\mathcal{H}^{d-1}dt\right| \\ \le&\|\eta\|_{C^0(U\times(0,T))}\left|\int_0^T \int_{M_t} (h \cdot v)\, d\mathcal{H}^{d-1}dt\right| \\ \le& C_T\|\eta\|_{C^0(U\times(0,T))}, \end{aligned} \tag{3.3}$$

where

$$\frac{d}{dt}\int_{M_t} \eta\, d\mathcal{H}^{d-1} = \int_{M_t} (-\eta h + \nabla\eta) \cdot v + \partial_t\eta\, d\mathcal{H}^{d-1} \tag{3.4}$$

and $\eta(\cdot,0) \equiv \eta(\cdot,T) \equiv 0$ are used. Therefore $\{\mu_t\}_{t\in(0,T)}$ is the L^2-flow with the generalized velocity vector $v = h + f$. The formula (3.4) also relates to the definition of Brakke's mean curvature flow.

Let φ^ε be a solution for (2.16). We define a Radon measure μ_t^ε by

$$\mu_t^\varepsilon(\phi) := \frac{1}{\sigma}\int_\Omega \phi\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)dx$$

for any $\phi \in C_c(\Omega)$. Here $\sigma = \int_{-1}^1 \sqrt{2W(s)}\, ds$. The main result of this article is the following:

Theorem 3.9. Let $d = 2, 3$ and $U_0 \subset \Omega$ be an open set with C^1 boundary M_0. Assume that there exist $C_1 > 0$, $\delta \in (0,1)$ and $\omega > 0$ such that $\|g\|_{C^1(\Omega\times[0,\infty))} \le C_1$, $\sup_{\Omega\times[0,\infty)} |g-1| \le \delta$, and

$$\omega \le \int_{U_0} g(x,t)\,dx \le (1-\delta)|\Omega| - \omega \qquad \text{for any } t \ge 0.$$

Then there exists a family of functions $\{\varphi_0^{\varepsilon_i}\}_{i=1}^{\infty}$ with $\varepsilon^i \downarrow 0$ as $i \to \infty$ such that the following hold:

(a) Let φ^{ε_i} be a solution for (2.16) with initial data $\varphi_0^{\varepsilon_i}$ for $i \ge 1$. Then there exists $\psi \in BV_{loc}(\Omega \times [0,\infty)) \cap C^{\frac{1}{2}}_{loc}([0,\infty); L^1(\Omega))$ such that

(a1) $\psi(\cdot, 0) = \chi_{U_0}$ a.e. on Ω and $\varphi^{\varepsilon_i} \to 2\psi - 1$ in $L^1_{loc}(\Omega \times [0,\infty))$ and a.e. pointwise.

(a2) (Volume preserving property) $\psi(\cdot,t)$ is a characteristic function with

$$\int_\Omega \psi(x,t)g(x,t)\,dx = \int_\Omega \psi(x,0)g(x,0)\,dx$$

for any $t \in [0,\infty)$.

(b) There exists a family of $(d-1)$-rectifiable and integral Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ on Ω such that $\mu_t^{\varepsilon_i} \to \mu_t$ as Radon measures on Ω for any $t \in [0,\infty)$.

(c) There exists $\lambda \in L^2_{loc}(0,\infty)$ such that for any $T > 0$, we have

$$\lambda^{\varepsilon_i} \to \lambda \ \text{ weakly in } L^2(0,T).$$

(d) There exists $f \in L^2_{loc}(0,\infty; (L^2(\mu_t))^d)$ such that $\{\mu_t\}_{t\in(0,\infty)}$ is a L^2-flow with a generalized velocity vector

$$v = h + f$$

and v satisfies

$$\lim_{i\to\infty} \int_{\{|\nabla\varphi^{\varepsilon_i}(\cdot,t)|\neq 0\}\times(0,\infty)} \frac{-\partial_t\varphi^{\varepsilon_i}}{|\nabla\varphi^{\varepsilon_i}|}\frac{\nabla\varphi^{\varepsilon_i}}{|\nabla\varphi^{\varepsilon_i}|}\cdot\Phi\,d\mu_t^{\varepsilon_i}dt = \int_{\Omega\times(0,\infty)} v\cdot\Phi\,d\mu_t dt$$

for any $\Phi \in C_c(\Omega \times [0,\infty); \mathbb{R}^d)$. Moreover there exists a measurable function $\theta : \partial^*\{\psi = 1\} \to \mathbb{N}$ such that

$$v = h - \frac{1}{\theta}\lambda g n \qquad \mathcal{H}^d\text{-a.e. on } \partial^*\{\psi = 1\}, \tag{3.5}$$

where n is the inner unit normal vector of $\{\psi(\cdot,t) = 1\}$ on $\partial^*\{\psi(\cdot,t) = 1\}$.

4 Outline of the proof

The key estimate is the following:

Lemma 4.1. Let $T > 0$, $d \geq 2$ and g satisfy the assumptions of Theorem 3.9. Assume that there exists $D > 0$ such that $\mu_0^\varepsilon(\Omega) = \sigma E(\varphi_0^\varepsilon) \leq D$ for any $\varepsilon \in (0,1)$. Then there exist $c_1 = c_1(d, \omega, \delta, D, T) > 0$ and $\epsilon_1 = \epsilon_1(d, \omega, \delta, D, T) \in (0,1)$ such that

$$\sup_{\varepsilon\in(0,\epsilon_1),\ 0\leq t\leq T} \mu_t^\varepsilon(\Omega) + \sup_{\varepsilon\in(0,\epsilon_1)} \int_0^T |\lambda^\varepsilon(t)|^2\, dt \leq c_1. \tag{4.1}$$

Note that the existence of $D > 0$ is natural from $\mathcal{H}^{d-1}(M_0) < \infty$, and the boundedness of $\mu_t^\varepsilon(\Omega)$ is not clear (see (2.19)). The proof of Lemma 4.1 is similar to that in [3] and [19]. To show the existence of the L^2-flow, we use the following:

Theorem 4.2 ([14]). Let $d = 2, 3$ and φ^ε be a solution for the following equation:

$$\begin{cases} \varepsilon\partial_t\varphi^\varepsilon = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + f^\varepsilon, & (x,t) \in \Omega \times (0,\infty). \\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x \in \Omega. \end{cases} \tag{4.2}$$

We assume that there exists $\tilde{\epsilon} > 0$ such that

$$\sup_{\varepsilon\in(0,\tilde{\epsilon})} \Big(\mu_0^\varepsilon(\Omega) + \int_0^T \int_\Omega \frac{1}{\varepsilon}(f^\varepsilon)^2\, dxdt\Big) < \infty \tag{4.3}$$

for any $T > 0$. Then there exits a subsequence $\varepsilon \to 0$ such that the following hold:

1. There exists a family of $(d-1)$-integral Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ on Ω such that

   (a) $\mu^\varepsilon \to \mu$ as Radon measures on $\Omega \times [0,\infty)$, where $d\mu = d\mu_t dt$.

   (b) $\mu_t^\varepsilon \to \mu_t$ as Radon measures on Ω for all $t \in [0,\infty)$.

2. There exists $f \in L^2_{loc}(0,\infty; (L^2(\mu_t))^d)$ such that

$$\lim_{\varepsilon\to 0} \frac{1}{\sigma} \int_{\Omega\times(0,\infty)} -f^\varepsilon \nabla\varphi^\varepsilon \cdot \Phi\, dxdt = \int_{\Omega\times(0,\infty)} f \cdot \Phi\, d\mu \tag{4.4}$$

   for any $\Phi \in C_c(\Omega \times [0,\infty); \mathbb{R}^d)$.

3. $\{\mu_t\}_{t\in(0,\infty)}$ is an L^2-flow with a generalized velocity vector $v = h + f$ and

$$\lim_{\varepsilon\to 0} \int_{\Omega\times(0,\infty)} v^\varepsilon \cdot \Phi\, d\mu^\varepsilon = \int_{\Omega\times(0,\infty)} v \cdot \Phi\, d\mu$$

   for any $\Phi \in C_c(\Omega \times [0,\infty); \mathbb{R}^d)$, where h is the generalized mean curvature vector of μ_t and

$$v^\varepsilon = \begin{cases} \frac{-\partial_t\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|}\frac{\nabla\varphi^\varepsilon}{|\nabla\varphi^\varepsilon|} & \text{if } |\nabla\varphi^\varepsilon| \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the reason for the assumption for d is that the following results are used for the proof of [14]:

Theorem 4.3 ([16]). Let $d = 2, 3$, $U \subset \mathbb{R}^d$ be an open set and $\{\varepsilon_i\}_{i=1}^{\infty}$ be a positive sequence such that $\varepsilon_i \to 0$ as $i \to \infty$. Assume that $\varphi^i \in C^2(U)$ for any $i \geq 1$. Set $\mu^i(\phi) := \frac{1}{\sigma}\int_U \phi\Big(\frac{\varepsilon_i|\nabla\varphi^i|^2}{2} + \frac{W(\varphi^i)}{\varepsilon_i}\Big)\,dx$. Suppose that

$$\sup_{i\in\mathbb{N}} \mu^i(U) < \infty, \quad \sup_{i\in\mathbb{N}} \int_U \varepsilon_i\Big(\Delta\varphi^i - \frac{W'(\varphi^i)}{\varepsilon_i^2}\Big)^2\,dx < \infty$$

and

$$\mu^i \to \mu \quad \text{as Radon measures.}$$

Then μ is integral and for any $\phi \in C_c(U)$ we have

$$\int_U \phi\Big(\frac{\varepsilon_i|\nabla\varphi^i|^2}{2} - \frac{W(\varphi^i)}{\varepsilon_i}\Big)\,dx \to 0 \quad \text{as } i \to \infty.$$

Moreover

$$\int_U |h|^2\,d\mu \leq \frac{1}{\sigma}\liminf_{i\to\infty}\int_U \varepsilon_i\Big(\Delta\varphi^i - \frac{W'(\varphi^i)}{\varepsilon_i^2}\Big)^2\,dx,$$

where h is the generalized mean curvature vector of μ.

Proof of Theorem 3.9. For simplicity, we show Theorem 3.9 under the assumptions of Lemma 4.1. Set $f^\varepsilon := \lambda^\varepsilon g\sqrt{2W(\varphi^\varepsilon)}$. Then we have

$$\begin{aligned}\int_0^T\int_\Omega \frac{1}{\varepsilon}(f^\varepsilon)^2\,dxdt \leq& 2(1+\delta)^2\int_0^T(\lambda^\varepsilon)^2\int_\Omega\frac{W(\varphi^\varepsilon)}{\varepsilon}\,dxdt\\ \leq& 2(1+\delta)^2\sigma^{-1}c_1^2,\end{aligned}$$

where (4.1) and $\int_\Omega \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx \leq \sigma^{-1}\mu_t(\Omega)$ is used. Thus we obtain (4.3) and there exists a subsequence $\{\varepsilon_{i_j}\}_{j=1}^{\infty}$ such that the conclusions of Theorem 4.2 hold.

By an argument similar to that in [12, Theorem 4.7], we obtain (a1). By (a1) and (2.17), we have

$$\begin{aligned}\int_\Omega \psi(x,t)g(x,t)\,dx &= \sigma^{-1}\lim_{j\to\infty}\int_\Omega \tilde{k}(\varphi^{\varepsilon_{i_j}}(x,t))g(x,t)\,dx\\ &= \sigma^{-1}\lim_{j\to\infty}\int_\Omega \tilde{k}(\varphi^{\varepsilon_{i_j}}(x,0))g(x,0)\,dx = \int_\Omega \psi(x,0)g(x,0)\,dx,\end{aligned}$$

where

$$\lim_{j\to\infty}\tilde{k}(\varphi^{\varepsilon_{i_j}}) = \lim_{j\to\infty}\int_0^{\varphi^{\varepsilon_{i_j}}}\sqrt{2W(s)}\,ds + \int_0^1\sqrt{2W(s)}\,ds = \sigma\psi \quad \text{a.e. on } \Omega\times(0,\infty) \tag{4.5}$$

is used. Note that $\sigma = 2\int_0^1\sqrt{2W(s)}\,ds$. Thus we have (a2).

By Lemma 4.1, there exist $\lambda \in L^2_{loc}(0,\infty)$ and a subsequence $\{\varepsilon_{i_j}\}_{j=1}^{\infty}$ (denoted by the same index) such that (c) holds.

Finally, we show (3.5). By (4.4), for any $\Phi \in C^1_c(\Omega \times [0,\infty); \mathbb{R}^d)$ we compute that

$$\begin{aligned}\int_{\Omega\times(0,\infty)} f\cdot\Phi\, d\mu &= \lim_{j\to\infty}\frac{1}{\sigma}\int_{\Omega\times(0,\infty)} -\lambda^{\varepsilon_{i_j}} g\sqrt{2W(\varphi^{\varepsilon_{i_j}})}\nabla\varphi^{\varepsilon_{i_j}}\cdot\Phi\, dxdt\\ &= \lim_{j\to\infty}\frac{1}{\sigma}\int_{\Omega\times(0,\infty)} -\lambda^{\varepsilon_{i_j}} g\nabla\tilde{k}(\varphi^{\varepsilon_{i_j}})\cdot\Phi\, dxdt\\ &= \lim_{j\to\infty}\frac{1}{\sigma}\int_{\Omega\times(0,\infty)} \lambda^{\varepsilon_{i_j}}\tilde{k}(\varphi^{\varepsilon_{i_j}})\operatorname{div}(g\Phi)\, dxdt.\end{aligned}\tag{4.6}$$

By (4.5), (4.6), and the Radon-Nikodym theorem we have

$$\begin{aligned}&\int_{\Omega\times(0,\infty)} f\cdot\Phi\, d\mu = \int_0^\infty \lambda\int_\Omega \psi\operatorname{div}(g\Phi)\, dxdt\\ =& -\int_0^\infty\lambda\int_\Omega g\nu\cdot\Phi\, d\|\nabla\psi(\cdot,t)\|dt = \int_{\Omega\times(0,\infty)} -\lambda g\frac{d\|\nabla\psi(\cdot,t)\|}{d\mu_t}\nu\cdot\Phi\, d\mu\end{aligned}\tag{4.7}$$

for any $\Phi \in C^1_c(\Omega\times[0,\infty);\mathbb{R}^d)$, where $\nu(\cdot,t)$ is the inner normal vector of $\{x \in \Omega\,|\,\psi(x,t) = 1\}$ on $\partial^*\{x\in\Omega\,|\,\psi(x,t)=1\}$. Set $\theta : \partial^*\{(x,t)\in\Omega\times(0,\infty)\,|\,\psi(x,t) = 1\} \to (0,\infty)$ by $\theta := \left(\frac{d\|\nabla\psi(\cdot,t)\|}{d\mu_t}\right)^{-1}$. Note that μ_t is integral by Theorem 4.3. Thus $\theta \in \mathbb{N}$ $\mathcal{H}^d$-a.e. Hence we have (3.5). □

References

[1] Allard, W., *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.

[2] Brakke, K. A., *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, N.J., (1978).

[3] Bronsard, L. and Stoth, B., *Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation*, SIAM J. Math. Anal., **28** (1997), 769–807.

[4] Chen, X., Hilhorst, D. and Logak, E., *Mass conserving Allen-Cahn equation and volume preserving mean curvature flow*, Interfaces Free Bound., **12** (2010), 527–549.

[5] Evans, L. C. and Gariepy, R. F., *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL (1992).

[6] Federer, H., *Geometric Measure Theory*, Springer-Verlag, New York, (1969).

[7] Escher, J. and Simonett, G., *The volume preserving mean curvature flow near spheres*, Proc. Amer. Math. Soc., **126** (1998), 2789–2796.

[8] Gage, M., *On an area-preserving evolution equation for plane curves*, Nonlinear problems in geometry (Mobile, Ala., 1985), Contemp. Math., **51** (1986), 51–62.

[9] Golovaty, D., *The volume-preserving motion by mean curvature as an asymptotic limit of reaction-diffusion equations*, Quart. Appl. Math., **55** (1997), no.2, 243–298.

[10] Huisken, G., *The volume preserving mean curvature flow*, J. Reine Angew. Math., **382** (1987), 35–48.

[11] Ilmanen, T., *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom., **38** (1993), no. 2, 417–461.

[12] Liu, C., Sato, N. and Tonegawa, Y., *On the existence of mean curvature flow with transport term*, Interfaces Free Bound., **12** (2010), no.2, 251–277.

[13] Mugnai, L. and Röger, M., *The Allen-Cahn action functional in higher dimensions*, Interfaces Free Bound., **10** (2008), 45–78.

[14] Mugnai, L. and Röger, M., *Convergence of perturbed Allen-Cahn equations to forced mean curvature flow*, Indiana Univ. Math. J., **60** (2011), 41–75.

[15] Mugnai, L., Seis, C. and Spadaro, E., *Global solutions to the volume-preserving mean-curvature flow*, Calc. Var. Partial Differential Equations, **55** (2016), 55:18.

[16] Röger, M. and Schätzle, R., *On a modified conjecture of De Giorgi*, Math. Z., **254** (2006), 675–714.

[17] Rubinstein, J. and Sternberg, P., *Nonlocal reaction-diffusion equations and nucleation*, IMA Journal of Applied Mathematics, **48** (1992), 249–264.

[18] Simon, L., *Lectures on geometric measure theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. **3** (1983).

[19] Takasao, K., *Existence of weak solution for volume preserving mean curvature flow via phase field method*, to appear in Indiana University Mathematics Journal.

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